BIBLIOGRAPHY

1. Krasovskii, N. N., On the stabilization of systems in which noise is dependent on the magnitude of controlling. Izv. Akad. Nauk SSSR, Tekhnicheskaia kibernetika №2, 1965.
2. Doob, J. L., Stochastic Processes. J. Wiley and Sons, N. Y., 1953.
3. Letov, A. M., Analytical construction of regulators, IV. Avtomatika i telemekhanika, Vol. 22, №4, 1961.
4. Krasovskii, N. N. and Lidskii, E. A., Anaytical construction of regulators in systems with random properties, I. Statement of the problem, method of solution. Avtomatika i telemekhanika, Vol. 22, №9, 1961; II. Optimal control equations. Approximate method of solution. Avtomatika i telemekhanika Vol. 22, №10, 1961; III. Optimal control in linear systems. Minimum of the mean square error. Avtomatika i telemekhanika, Vol. 22, №11, 1961.
5. Wonham, W. N., Optimal stationary control of a linear system with state-dependent noise. SIAM J. Control., Vol. 5, №3, 1967.
6. Wonham, W. N., Lecture notes on stochastic control. Brown University, 1967.
7. Lur'e, A. I., Minimal quadratic quality criterion of control of a system. Izv. Akad Nauk SSSR, Tekhnicheskaia kibernetika, №4, 1963.
8. Priakhin, N. S., On the problem of analytic construction of regulators. Avtomatika i telemekhanika, Vol. 24, №9, 1963.
9. Nevel'son, M. B. and Khas'minskii, R. Z., Stability of a linear system with random disturbances of its parameters. PMM Vol. 30, №2, 1967.
10. Nevel'son, M. B. and Khas'minskii, R. Z., On the stability of stochastic systems. Problemy peredachi informatsii, Vol. 2, №3, 1966.

Translated by L. K.

ON A CRITERION OF STABILITY OF SOLUTIONS OF AN N-TH ORDER LINEAR DIFFERENTIAL EQUATION

PMM Vol. 33, №3, 1969, pp. 578-579
V. N. BEREZHNOI and Iu. S. KOLESOV
(Voronezh)
(Received December 17, 1968)

Using the methods of the theory of cones we establish a sufficient condition of stability of solutions of an n-th order linear differential equation.

Let us consider the following linear differential equation:

$$\frac{d^n x}{dt^n} + p_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + p_n(t)\, x = 0 \tag{1}$$

where $p_i(t)$ $(i = 1, \ldots, n;\ t_0 \leqslant t < \infty)$ are continuous functions. We shall indicate one criterion of the stability of solutions of (1) in terms of its characteristic polynomial

$$P(t, \lambda) = \lambda^n + p_1(t)\lambda^{n-1} + \ldots + p_n(t) \tag{2}$$

We will use certain concepts of the theory of cones [1, 2].

Let us write Eq. (1) in the form of a first order equation in an n-dimensional Euclidean space R^n

$$\frac{du}{dt} = Q(t)\,u, \qquad u = \left(x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right) \tag{3}$$

$$Q(t) = \begin{Vmatrix} 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ -p_n(t) & -p_{n-1}(t) & -p_{n-2}(t) & \ldots & -p_1(t) \end{Vmatrix}$$

denoting its fundamental matrix by $U(t, s)$ $(U(s, s) = I)$.

An important part will be played by the cone K_0 which is defined as follows. Let $\lambda_1, \ldots, \lambda_{n-1}$ be certain constants and let us consider the polynomials

$$Q_k(\lambda) = (\lambda - \lambda_1) \ldots (\lambda - \lambda_{k-1}) = \lambda^{k-1} + a_{k,k-1}\lambda^{k-2} + \ldots + a_{k1} \quad (k = 2, \ldots, n) \tag{4}$$

constructing the matrix

$$A = \begin{Vmatrix} 1 & 0 & 0 & \ldots & 0 \\ a_{21} & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & 1 \end{Vmatrix}$$

The cone K_0 will have the form

$$K_0 = \{u : Au^+ \geqslant 0\} \tag{5}$$

Inequality $Au^+ \geqslant 0$ means that the vector Au belongs to the cone K_+ of vectors with nonnegative coordinates. Semiorderliness generated by the cone K_0 will be denoted by $\overset{\circ}{\geqslant}$ or $\overset{\circ}{\leqslant}$. We now introduce the following notation:

$$\Delta^\circ(t, \lambda_1) = P(t, \lambda_1), \quad \Delta^1(t, \lambda_1, \lambda_2) = \frac{P(t, \lambda_1) - P(t, \lambda_2)}{\lambda_1 - \lambda_2}, \ldots$$

$$\Delta^{n-2}(t, \lambda_1, \ldots, \lambda_{n-1}) = \frac{\Delta^{n-3}(t, \lambda_1, \ldots, \lambda_{n-2}) - \Delta^{n-3}(t, \lambda_2, \ldots, \lambda_{n-1})}{\lambda_1 - \lambda_{n-1}}$$

If some of λ_i are equal to each other, the formulas given above should be interpreted in their limiting form, i. e. a small perturbation should be applied to λ_i followed by the passage to the limit.

Lemma. Let constants $\lambda_1, \ldots, \lambda_{n-1}$ exist such that

$$\Delta^k(t, \lambda_1, \ldots \lambda_{k+1}) \leqslant 0 \qquad (k = 0, \ldots, n-2;\ t_0 \leqslant t < \infty) \tag{6}$$

Then the cone K_0 defined by the formula (5) remains invariant under the action of the operator $U(t, s)$ $(t_0 \leqslant s \leqslant t < \infty)$.

Proof. Using the substitution

$$v = Au \tag{7}$$

Eq. (3) takes the form

$$\frac{dv}{dt} = AQ(t)A^{-1}v \tag{8}$$

Direct computation shows that

$$AQ(t)A^{-1} =$$

$$= \begin{Vmatrix} \lambda_1 & 1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & & 1 \\ -\Delta^\circ(t, \lambda_1) & -\Delta^1(t, \lambda_1, \lambda_2) & -\Delta^2(t, \lambda_1, \lambda_2) & \ldots & -p_1(t) - \lambda_1 - \ldots - \lambda_{n-1} \end{Vmatrix}$$

The above and the inequality (6) imply that the fundamental matrix $V(t, s)$ $(t_0 \leqslant s \leqslant t < \infty)$ of Eq. (8) transforms the cone K_+ into itself. Since (7) transforms the cone K_0

into the cone K_+, therefore the matrix $U(t, s)$ $(t_0 \leqslant s \leqslant t < \infty)$ transforms the cone K_0 into itself. The Lemma is proved.

Theorem 1. Let a constant λ_0 exist under the conditions of the Lemma, such that

$$\lambda_0 > \max \Lambda_i \qquad (1 \leqslant i \leqslant n-1) \tag{9}$$

$$P(t, \lambda_0) \geqslant 0 \qquad (t_0 \leqslant t < \infty) \tag{10}$$

Then the following estimate holds:

$$|U(t, s)| \leqslant M e^{\lambda_0 (t-s)} \qquad (t_0 \leqslant s \leqslant t < \infty) \tag{11}$$

where M is some number.

Proof. We set $u_0 = (1, \lambda_0, \ldots, \lambda_0^{n-1})$. It is easy to see that $Au_0 = (1, Q_2(\lambda_0), \ldots, Q_n(\lambda_0))$ (where $Q_k(\lambda)$ are the polynomials (4)). It follows that the vector u_0 will lie within the cone K_0. Therefore we can introduce the following equivalent norm (so called u_0-norm [2]) in the space R^n:

$$|u|_{u0} = \min \alpha \qquad (-\alpha u_0 \leqslant^{\circ} u \leqslant^{\circ} \alpha u_0)$$

We will analyze the function $u_0(t) = e^{\lambda_0 (t-s)}$. From the inequality (10) it follows that:

$$\frac{du_0(t)^{\circ}}{dt} \geqslant Q(t)\, u_0(t) \qquad (t \geqslant s \geqslant t_0) \tag{12}$$

Let now $-u_0 \leqslant^{\circ} u \leqslant^{\circ} u_0$. Then from (12) it follows that:

$$-e^{\lambda_0(t-s)} u_0 \leqslant^{\circ} U(t, s)\, u \leqslant^{\circ} e^{\lambda_0(t-s)} u_0 \qquad (t \geqslant s \geqslant t_0)$$

and

$$|U(t, s)|_{u_0} \leqslant e^{\lambda_0(t-s)} \qquad (t \geqslant s \geqslant t_0) \tag{13}$$

Inequality (13) proves the inequality (11). The theorem is proved.

Corollary. Let $\lambda_0 < 0$ under the condition of Theorem 1, consequently solutions of (1) are exponentially stable.

Theorem 2. Let the inequality (10) under the conditions of Theorem 1 be replaced by

$$P(t, \lambda_0) \leqslant 0 \qquad (t_0 \leqslant t < \infty, \ \lambda_0 > 0)$$

Then the zero solution of (1) is unstable.

The proof of Theorem 2 which resembles that of Theorem 1, is omitted.

BIBLIOGRAPHY

1. Krein, M. G. and Rutman, M. A., Linear operators leaving a cone invariant in the Banach space. Usp. matem. nauk, Vol. 3, №1, 1948.
2. Krasnosel'skii, M. A., Positive Solutions of Operator Equations. M. Fizmatgiz, 1962.

Translated by L. K.

ON THE INSTABILITY OF A PLANE TANGENTIAL DISCONTINUITY

PMM Vol. 33, №3, 1969, pp. 580-581

N. G. KIKINA and D. G. SANNIKOV
(Moscow)

(Received June 13, 1968)

The problem of instability of a plane tangential discontinuity which was already considered in [1, 2], is solved here in connection with the problem on reflection of plane monochromatic waves from a surface of discontinuity. Dependence of the decremental